Not Really a Knot Talk (Part 1)

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GARTS

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Outline

1. Monoidal Categories

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2. Ribbon Categories
    - Braidings, twistings, and compatible "duals"

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    - Braidings, twistings, and compatible "duals"
3. Trace in a Ribbon Category

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2. Ribbon Categories
   - Braidings, twistings, and compatible "duals"
3. Trace in a Ribbon Category
4. Graphical Notation

The categories Rep($\mathfrak{g}$) and Rep(G)

Recall for U and V fin.-dim. representations (over $\mathbb{C}$) of $\mathfrak{g}$ (resp. G) one defines a $\mathfrak{g}$-module (resp. G-module) structure on $U \otimes V$, $\mathbb{C}$, and U^* by the following actions

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These formulas are coming from corresponding Hopf algebras: the universal enveloping algebra $U(\mathfrak{g})$ and the group algebra $\mathbb{C}[G]$, respectively.

The categories Rep($\mathfrak{g}$) and Rep(G) (cont.)

$$U \otimes V \Longleftrightarrow \Delta$$

The categories Rep($\mathfrak{g}$) and Rep(G) (cont.)

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$$\mathbb{C} \Longleftrightarrow \varepsilon$$

The categories Rep($\mathfrak{g}$) and Rep(G) (cont.)

$$U \otimes V \Longleftrightarrow \Delta$$
$$\mathbb{C} \Longleftrightarrow \varepsilon$$
$$\mathrm{Hom}(U, \mathbb{C}) =: U^* \Longleftrightarrow S$$

Comultiplication, Counit, and Antipode

Coassociativity

$$
\begin{tikzcd}
(H \otimes H) \otimes H & & H \otimes (H \otimes H) \arrow[ll, "\alpha^{-1}_{H,H,H}"'] \\
H \otimes H \arrow[u, "\Delta \otimes \mathrm{id}_H"] & & H \otimes H \arrow[u, "\mathrm{id}_H \otimes \Delta"'] \\
& H \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
$$

Counitality

$$
\begin{tikzcd}
H \otimes \mathbb{C} & & \mathbb{C} \otimes H \\
& H \arrow[ul, "\rho_H^{-1}"'] \arrow[ur, "\lambda_H^{-1}"] \arrow[dl, "\Delta"] \arrow[dr, "\Delta"'] & \\
H \otimes H \arrow[uu, "\mathrm{id}_H \otimes \varepsilon"] & & H \otimes H \arrow[uu, "\varepsilon \otimes \mathrm{id}_H"']
\end{tikzcd}
$$

Comultiplication, Counit, and Antipode (cont.)

Antipode

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\iota\circ\varepsilon"'] & H \otimes H \arrow[d, "\mathrm{id}\otimes S"] \\ H & H \otimes H \arrow[l, "\mu"] \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\iota\circ\varepsilon"'] & H \otimes H \arrow[d, "S\otimes\mathrm{id}"] \\ H & H \otimes H \arrow[l, "\mu"] \end{tikzcd}$$

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Let U, V, and W be vector spaces. Then there is a natural isomorphism

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Note that theses spaces are isomorphic and not equal. Moreover, this isomorphism allows us to move a pair of parentheses in our tensor product and get isomorphic objects.

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What if we have nested parentheses?

Associativity of the Tensor Product (cont.)

$$\begin{tikzcd}[column sep=small]
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "\alpha_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "\alpha_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "\alpha_{W, X \otimes Y, Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\alpha_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes \alpha_{X,Y,Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

Left- and Right-Unitality of $\mathbb{C}$

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Moreover, these maps satisfy the following commutative diagram

$$\begin{tikzcd}
(U \otimes \mathbb{C}) \otimes V \arrow[rr, "\alpha_{U,\mathbb{C},V}"] \arrow[dr, "\rho_U \otimes \mathrm{id}_V"'] & & U \otimes (\mathbb{C} \otimes V) \arrow[dl, "\mathrm{id}_U \otimes \lambda_V"] \\
& U \otimes V &
\end{tikzcd}$$

Unacknowledged Proposition

Proposition

The maps $\alpha_{U,V,W}, \lambda_U$, and ρ_U are isomorphisms of $\mathfrak{g}$-modules (resp. G-modules) for all $\mathfrak{g}$-modules (resp. G-modules) U, V, and W.

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The maps $\alpha_{U,V,W}, \lambda_U$, and ρ_U are isomorphisms of $\mathfrak{g}$-modules (resp. G-modules) for all $\mathfrak{g}$-modules (resp. G-modules) U, V, and W.

(Sketch of) Proof.

That $\alpha_{U,V,W}$ is an isomorphism of modules is equivalent to the coassociativity of Δ. That λ_U and ρ_U are isomorphisms of modules is equivalent to the counit laws. $\square$

Monoidal Categories

Definition

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$$\alpha : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \to \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)$$

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$$\alpha : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \to \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)$$
$$\lambda : \mathbb{1} \otimes - \to \mathrm{id}_{\mathcal{C}}$$

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$$\begin{aligned} \alpha &: \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \to \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes) \\ \lambda &: \mathbb{1} \otimes - \to \mathrm{id}_{\mathcal{C}} \\ \rho &: - \otimes \mathbb{1} \to \mathrm{id}_{\mathcal{C}} \end{aligned}$$

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$$\begin{aligned}\alpha :& \otimes \circ(\otimes \times \mathrm{id}_{\mathcal{C}}) \to \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)\\ \lambda :& \mathbb{1} \otimes - \to \mathrm{id}_{\mathcal{C}}\\ \rho :& - \otimes \mathbb{1} \to \mathrm{id}_{\mathcal{C}}\end{aligned}$$

called the associator, the left-unitor, and the right-unitor, respectively, such that their components satisfy the following commutative diagrams:

Pentagon Axiom

$$
\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "\alpha_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "\alpha_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "\alpha_{W, X \otimes Y, Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\alpha_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes \alpha_{X,Y,Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}
$$

Triangle Axiom

$$
\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "\rho_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \lambda_Y"] \\
 & X \otimes Y &
\end{tikzcd}
$$

We Know Some Monoidal Categories!

We Know Some Monoidal Categories!

Proposition

kVect, R-mod, Rep(G), Rep$(\mathfrak{g})$, and Rep$(U_q(\mathfrak{g}))$ are monoidal categories.

Strict Monoidal Categories

Definition

A monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is called strict if α, λ, and ρ are the identity natural isomorphisms. In particular, in such a category we have

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z), \quad \mathbb{1} \otimes X = X = X \otimes \mathbb{1}$$

For every object X, Y, and Z in $\mathcal{C}$.

MacLane's Strictness Theorem

Theorem

Every monoidal category is monoidally equivalent to a strict monoidal category.

MacLane's Strictness Theorem

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Proof.

See section 2.8 of **Tensor Categories** by P. Etingof, et. al. $\square$

Braided Monoidal Category

Definition

A braided monoidal category is a heptuple $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \beta)$ such that $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category, and $\beta : \otimes \to \otimes \circ T$ is a natural isomorphism, where

$$T : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}, \quad (X, Y) \mapsto (Y, X),$$

such that the components of β satisfy the hexagon axioms.

Braided Monoidal Category (cont.)

Hexagon Axioms

$$
\begin{tikzcd}
 & (V \otimes U) \otimes W \arrow[r, "\alpha_{V,U,W}"] & V \otimes (U \otimes W) \arrow[dr, "\mathrm{id}_V \otimes \beta_{U,W}"] & \\
(U \otimes V) \otimes W \arrow[ur, "\beta_{U,V} \otimes \mathrm{id}_W"] \arrow[dr, "\alpha_{U,V,W}"'] & & & V \otimes (W \otimes U) \\
 & U \otimes (V \otimes W) \arrow[r, "\beta_{U,V\otimes W}"'] & (V \otimes W) \otimes U \arrow[ur, "\alpha_{V,W,U}"'] &
\end{tikzcd}
$$

and

Braided Monoidal Category (cont.)

Hexagon Axioms

$$
\begin{tikzcd}
 & (V \otimes U) \otimes W \arrow[r, "\alpha_{V,U,W}"] & V \otimes (U \otimes W) \arrow[dr, "\mathrm{id}_V \otimes \beta_{U,W}"] & \\
(U \otimes V) \otimes W \arrow[ur, "\beta_{U,V} \otimes \mathrm{id}_W"] \arrow[dr, "\alpha_{U,V,W}"'] & & & V \otimes (W \otimes U) \\
 & U \otimes (V \otimes W) \arrow[r, "\beta_{U,V \otimes W}"'] & (V \otimes W) \otimes U \arrow[ur, "\alpha_{V,W,U}"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & U \otimes (W \otimes V) \arrow[r, "\alpha^{-1}_{U,W,V}"] & (U \otimes W) \otimes V \arrow[dr, "\beta_{U,W} \otimes \mathrm{id}_V"] & \\
U \otimes (V \otimes W) \arrow[ur, "\mathrm{id}_U \otimes \beta_{V,W}"] \arrow[dr, "\alpha^{-1}_{U,V,W}"'] & & & (W \otimes U) \otimes V \\
 & (U \otimes V) \otimes W \arrow[r, "\beta_{U \otimes V,W}"'] & W \otimes (U \otimes V) \arrow[ur, "\alpha^{-1}_{W,U,V}"'] &
\end{tikzcd}
$$

Symmetric Monoidal Category

Definition

A symmetric monoidal category is a braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \beta)$ such that

$$\beta_{Y,X} \circ \beta_{X,Y} = \mathrm{id}_{X \otimes Y}$$

for all objects X and Y of $\mathcal{C}$.

Twisting

Definition

A twisting for a braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \beta)$ is a natural isomorphism $\theta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ whose components satisfy the following commutative diagram

$$\begin{tikzcd} U \otimes V \arrow[r, "\theta_{U\otimes V}"] \arrow[d, "\theta_U \otimes \theta_V"'] & U \otimes V \\ U \otimes V \arrow[r, "\beta_{U,V}"'] & V \otimes U \arrow[u, "\beta_{V,U}"'] \end{tikzcd}$$

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$$\begin{tikzcd}
U \otimes V \arrow[r, "\theta_{U\otimes V}"] \arrow[d, "\theta_U \otimes \theta_V"'] & U \otimes V \\
U \otimes V \arrow[r, "\beta_{U,V}"'] & V \otimes U \arrow[u, "\beta_{V,U}"']
\end{tikzcd}$$

Formulaically, the diagram reads

$$\theta_{U\otimes V} = \beta_{V,U} \circ \beta_{U,V} \circ \theta_U \otimes \theta_V$$

Remark: This is sometimes called a balanced monoidal category.

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where in the definition of b_V, the v_j's and f_j's are dual bases of V and V^* respectively such that $f_j(v_i) = \delta_{ij}$. These maps are called the evaluation and coevaluation maps, respectively.

Vector Space Duals (cont.)

$$\begin{tikzcd}
(V \otimes V^*) \otimes V \arrow[rr, "\alpha_{V,V^*,V}"] & & V \otimes (V^* \otimes V) \arrow[d, "\mathrm{id}_V \otimes d_V"] \\
\mathbb{C} \otimes V \arrow[u, "b_V \otimes \mathrm{id}_V"] \arrow[dr, "\lambda_V"'] & & V \otimes \mathbb{C} \\
& V \arrow[ur, "\rho_V^{-1}"'] &
\end{tikzcd}$$

$$\Rightarrow (\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V$$

and

Vector Space Duals (cont.)

$$
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& V \arrow[ur, "\rho_V^{-1}"'] &
\end{tikzcd}
\Rightarrow (\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V
$$

and

$$
\begin{tikzcd}
V^* \otimes (V \otimes V^*) \arrow[rr, "\alpha^{-1}_{V^*,V,V^*}"] & & (V^* \otimes V) \otimes V^* \arrow[d, "d_V \otimes \mathrm{id}_{V^*}"] \\
V^* \otimes \mathbb{C} \arrow[u, "\mathrm{id}_{V^*} \otimes b_V"] \arrow[dr, "\rho_{V^*}"'] & & \mathbb{C} \otimes V^* \\
& V^* \arrow[ur, "\lambda^{-1}_{V^*}"'] &
\end{tikzcd}
\Rightarrow (d_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}
$$

Monoidal Categories with (Left-)Duals

Definition

We say that a monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda)$ has (left-)duals if for every object V of $\mathcal{C}$ there exists another object V^* and morphisms $d_V : V^* \otimes V \to \mathbb{1}$ and $b_V : \mathbb{1} \to V \otimes V^*$ such that

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$$(\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V$$
$$(d_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$$

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$$(\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V$$
$$(d_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$$

Note: As before, the "structure natural isomorphisms" are suppressed.

Monoidal Categories with (Left-)Duals (cont.)

Proposition

The categories $\mathbf{kVect}$, $\mathbf{Rep}(\mathfrak{g})$, $\mathbf{Rep}(\mathbf{G})$, *and* $\mathbf{Rep}(\mathbf{U_q}(\mathfrak{g}))$ *are monoidal categories with duals.*

Monoidal Categories with (Left-)Duals (cont.)

Proposition

The categories **kVect**, $\mathbf{Rep}(\mathfrak{g})$, $\mathbf{Rep}(\mathbf{G})$, *and* $\mathbf{Rep}(\mathbf{U_q}(\mathfrak{g}))$ *are monoidal categories with duals.*

(Sketch of) Proof.

It is a tedious yet insightful check that $(\mathrm{Hom}(V, \mathbb{C}), d_V, b_V)$ with d_V and b_V the aforementioned linear maps is a left dual to V in the respective categories. Specifically, that d_V and b_V are maps preserving the respective module structures follows from properties of the corresponding antipode. □

Ribbon Categories

Definition

A ribbon category is a balanced monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda, \beta, \theta)$ with duals such that for every object V of $\mathcal{C}$

$$(\theta_V \otimes \mathrm{id}_{V^*}) \circ b_V = (\mathrm{id}_V \otimes \theta_{V^*}) \circ b_V$$

Ribbon Categories (cont.)

- **kVect**

Ribbon Categories (cont.)

- **kVect** ... with boring transposition and trivial twisting

Ribbon Categories (cont.)

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Ribbon Categories (cont.)

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Ribbon Categories (cont.)

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- **Rep**($\mathfrak{g}$) ... with boring transposition and trivial twisting
- **Rep**(**G**)

Ribbon Categories (cont.)

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- $\mathbf{Rep}(\mathfrak{g})$ ... with boring transposition and trivial twisting
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Ribbon Categories (cont.)

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- **Rep**$(\mathfrak{g})$ ... with boring transposition and trivial twisting
- **Rep**$(\mathbf{G})$ ... with boring transposition and trivial twisting
- **Rep**$(\mathbf{U_q}(\mathfrak{g}))$

Ribbon Categories (cont.)

- **kVect** ... with boring transposition and trivial twisting
- **Rep**($\mathfrak{g}$) ... with boring transposition and trivial twisting
- **Rep**(**G**) ... with boring transposition and trivial twisting
- **Rep**($\mathbf{U_q}$($\mathfrak{g}$)) ... with intricate braiding and highly nontrivial twisting

Traces and Dimensions

Definition

For an endomorphism $f : V \to V$ of an object V in a ribbon category $\mathcal{C}$, we define its trace $\mathrm{tr}(f) \in \mathrm{End}(\mathbb{1})$ to be the composition

Traces and Dimensions

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For an endomorphism $f : V \to V$ of an object V in a ribbon category $\mathcal{C}$, we define its trace $\mathrm{tr}(f) \in \mathrm{End}(\mathbb{1})$ to be the composition

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Note: This agrees with the usual notion of trace and dimension in our working examples of ribbon categories.

Lemma

1. *For any morphisms* $f : V \to W$ *and* $g : W \to V$, *we have* $tr(f \circ g) = tr(g \circ f)$.
2. *For any endomorphisms* f *and* g, *we have* $tr(f \otimes g) = tr(f) \circ tr(g)$.
3. *For any endomorphism* $k \in End(\mathbb{1})$, *we have* $tr(k) = k$.

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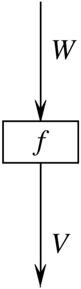

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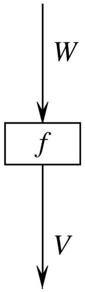

We call V and W the colors of the arrows, and f is the color of the box.

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More generally, a morphism $f : V_1 \otimes \cdots \otimes V_m \to W_1 \otimes \cdots \otimes W_n$ will be represented by the following figure

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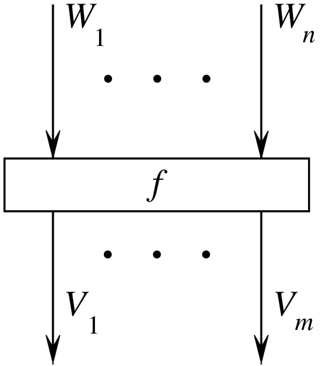

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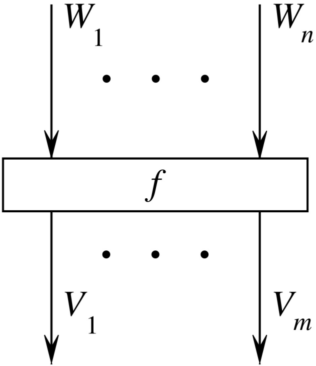

Note: If $m, n = 0$, then one interprets the corresponding empty tensor product as equal to the unit object $\mathbb{1}$ and no arrows are drawn.

Graphical Notation for Morphisms (cont.)

We will also use vertical arrows oriented upwards with the convention that the coloring of the arrow is in fact the dual object e.g. the morphism $f : V^* \to W^*$ may be drawn in four equivalent ways

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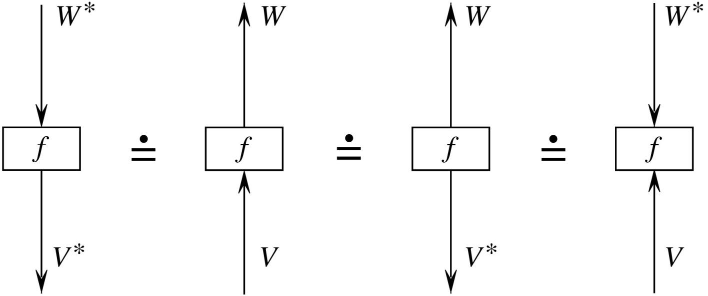

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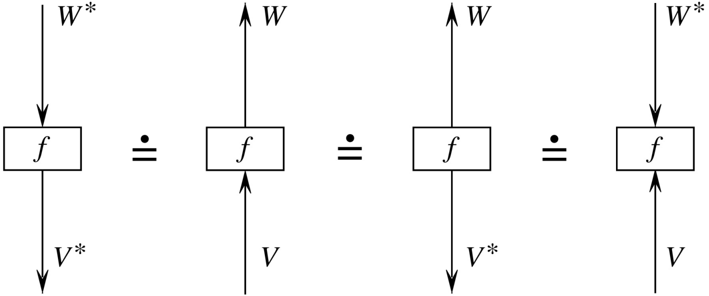

Where the symbol $\doteq$ denotes equality of the corresponding morphisms in $\mathscr{V}$.

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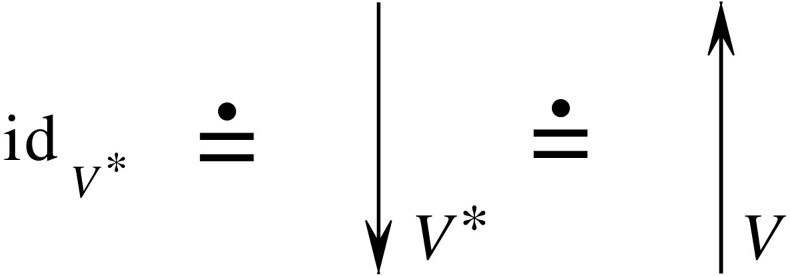

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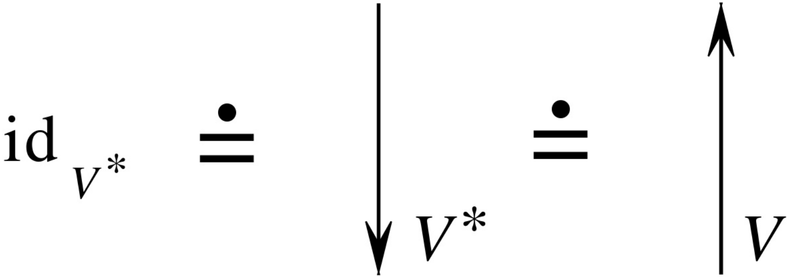

By convention, the empty picture is the identity endomorphism of $\mathbb{1}$.

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- The composition of two morphisms f and g is obtained by vertically stacking a picture of f on top of a picture of g.

Graphical Notation for Morphisms (cont.)

For example, for morphisms $f : V \to W$ and $g : V' \to W'$, the identities

$$(f \otimes \mathrm{id}_{W'}) \circ (\mathrm{id}_V \otimes g) = f \otimes g = (\mathrm{id}_{V'} \otimes g) \circ (f \otimes \mathrm{id}_W)$$

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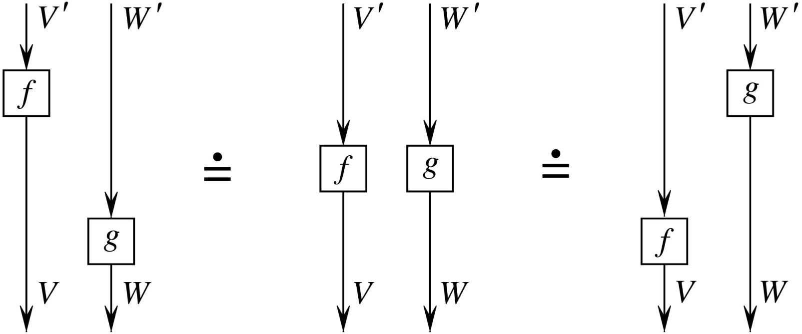

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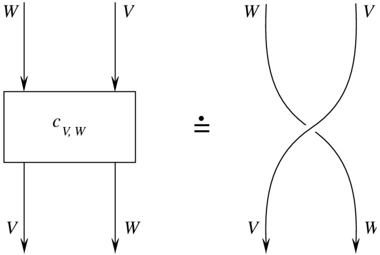
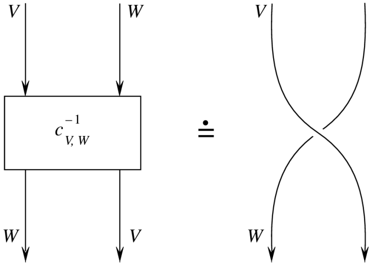


$\doteq$



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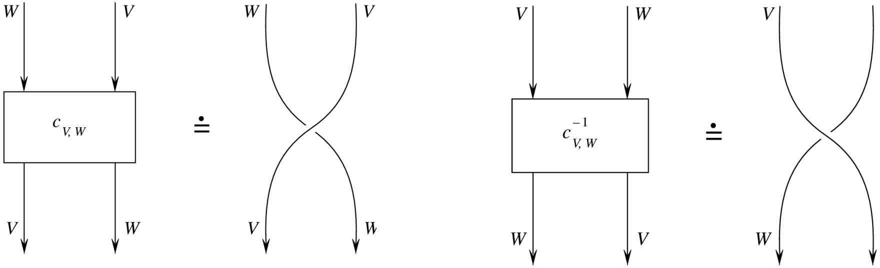

Note: Colors of arrows do not change when going through a crossing. They may change only when arrows hit coupons.

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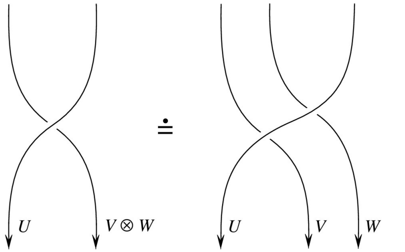

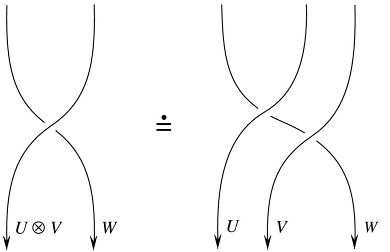

$\beta_{U,V\otimes W} = (\mathrm{id}_V \otimes \beta_{U,W}) \circ (\beta_{U,V} \otimes \mathrm{id}_W),\quad \beta_{U\otimes V,W} = (\beta_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \beta_{V,W})$

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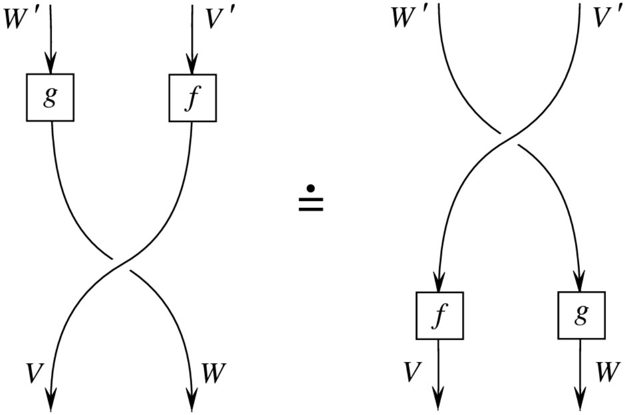

$$(g \otimes f) \circ \beta_{V,W} = \beta_{V',W'} \circ (f \otimes g)$$

Graphical Notation for Morphisms (cont.)

For every object V, the maps $b_V : \mathbb{1} \to V \otimes V^*$ and $d_V : V^* \otimes V \to \mathbb{1}$ will be represented by the following right-oriented cup and cap diagrams, respectively:

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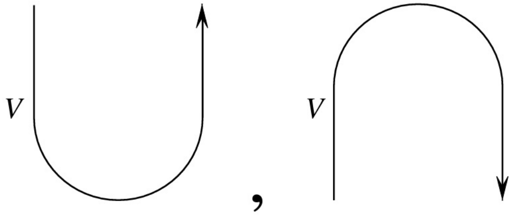

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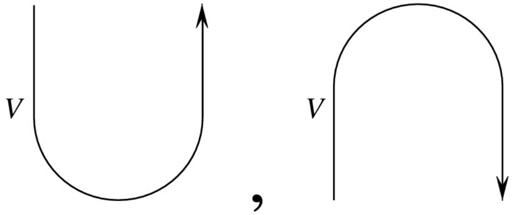

Where recall that the unit object has no arrows drawn to it or from it, and that upward oriented arrows represent duals.

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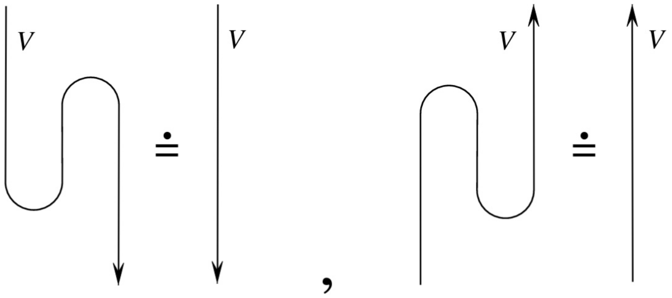

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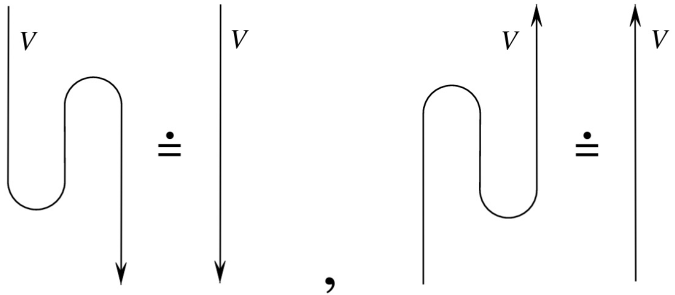

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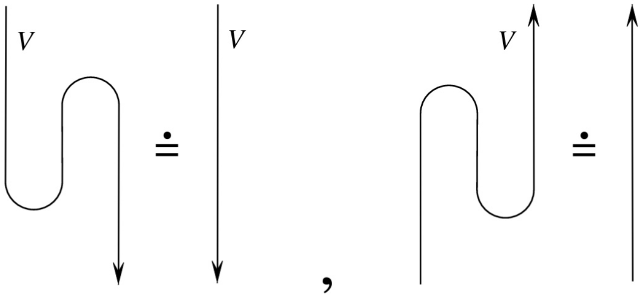

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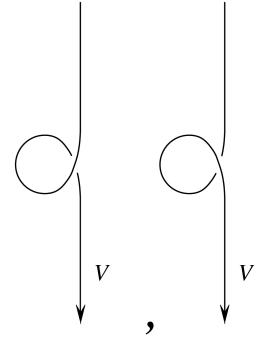

θ_V $\qquad\qquad$ θ_V^{-1}

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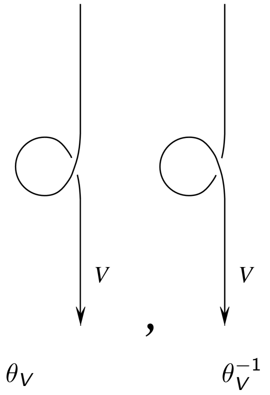

These are sometimes called positive and negative twists, respectively.

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- One might wish to straighten out the twisting i.e. perform a type 1 Reidemeister move.
- For those familiar with knots, we will come to see next week that our diagrams are, however, compatible with the type 2 and type 3 Reidemesiter moves, and the modified type 1 Reidemesiter move.

Thank You!

Thank You! **applause**